Path integral formulation of QFT (III)

■ Path integral for fermions

- Peskin-Schroeder 9.5
- Schwartz 14.6
- Cheng-Li 1.3

In the following we introduce a set of rules to deal with what is called "Grassman numbers". The reason is that we want to perform the path integral on fermions, which anti-commute. The set of rules are justified a posteriori, when we compare the results from the fermionic path integral with the ones obtained from canonical quantization for simple enough theories.

The starting point is to introduce the Grassman variables. Say θ and η are

Grassman. Then, by definition anticommute

$$\theta\eta = -\eta\theta$$

This implies

$$\theta^2 = \eta^2 = 0$$

But also that

$$(\theta\eta)(\theta'\eta') = +(\theta'\eta')(\theta\eta)$$

so $\theta\eta$ behaves like a normal number.

If

$$A = B + C\theta,$$

if $A \in \mathbb{C}$, then $B \in \mathbb{C}$ & C is Grassman. If A is Grassman, then B is also while $C \in \mathbb{C}$.

- We will be interested in integrating over θ. So we want to define

$$\int d\theta\, f(\theta).$$

First, note that $\theta^2 = 0$ implies that the Taylor expansion truncates, and the only functions are linear:

$$f(\theta) = A + B\theta \qquad \forall\, B \in \mathbb{C},\ A \in \text{"Grass."}$$

The integral is <u>defined</u> to be

$$\int d\theta\, f(\theta) = \int d\theta\,[A + B\theta] \equiv B.$$

A property that this definition has is that it preserves the shift-invariant property of regular integrals:

$$\int_{-\infty}^{\infty} dx\, f(x) = \int_{-\infty}^{\infty} dx\, f(x+a)$$

Indeed,

$$\int d\theta\, f(\theta+\eta) = \int d\theta\,\left(A + B\theta + B\eta\right) = B = \int d\theta\, f(\theta)$$

This shift is essential in order to do path integrals.

- We can define multiple integrals:

$$\int d\theta\, d\eta\, f(\theta,\eta) = \int d\theta\, d\eta\,[A + B_1\theta + B_2\eta + C\eta\theta]$$

$$\equiv C$$

This is a convention in the ordering. We do integrals by pairing measure & integrand

$$\int d\theta\, d\eta\, \eta\theta = \int d\theta\, \theta = 1 = -\int d\theta\, d\eta\, \theta\eta$$

In general,

$$\int d\theta_1 \cdots d\theta_k \, f(\vec{\theta}) = \int d\theta_1 \cdots d\theta_k \, C \, \theta_k \cdots \theta_1 = C$$

Funny enough, this definition looks like a reasonable definition for the derivative:

$$\frac{d}{d\theta} f(\theta) = \frac{d}{d\theta}(A + B\theta) = B = \int d\theta \, f(\theta)$$

- Change of variables works <u>opposite</u> to the standard integrals. For instance,

$$\int_{-\infty}^{\infty} dx \, f(2x) = \frac{1}{2} \int_{-\infty}^{\infty} dy \, f(y)$$

while

$$\int d\theta \, f(2\theta) = 2 \int d\theta \, f(\theta)$$

(but notice that $\frac{\partial f(2x)}{\partial x} = 2 \frac{\partial f(x)}{\partial x}$)

This is general. While one has the familiar relation

$$\int dx' f(x') = \int dx \left(\frac{dx'}{dx}\right) f(x'(x))$$

for Grassmanian, $\theta \to \theta' = a + b\theta$,

$$\left.\begin{array}{l} \int d\theta' f(\theta') = B \\ \int d\theta \, f(\theta') = bB \end{array}\right\} \quad \int d\theta' f(\theta') = \int d\theta \left(\frac{d\theta'}{d\theta}\right) f(\theta'(\theta))$$

In the multivariate case,

$$(\theta_1 \cdots \theta_K)\big|_{\vec{\theta} = L\cdot\vec{\theta}} = \sum_{i_1,\dots,i_K} L_{1i_1} \cdots L_{Ki_K} \underbrace{\theta_{i_1} \cdots \theta_{i_K}}_{\epsilon_{i_1 \cdots i_K}\theta_1 \cdots \theta_K}$$

$$= \det L \cdot \theta_1 \cdots \theta_K$$

Therefore,

$$\int d^n\theta\, f(L\vec{\theta}) = \det L \int d^n\theta\, f(\vec{\theta})$$

↳ instead of the usual $1/\det L$

- Conjugation

One needs to define a notion of complex conj. for Grassman variables. C.C. will gives a notion to assess whether the action is real and therefore unitary.

The "*" operation connects two distinct variables, that we call θ & θ^*,

$$(\theta)^* \equiv \theta^* \quad ; \quad (\theta^*)^* = \theta$$

For Grassman × number,

$$(a\theta)^* = a^*\theta^* = \theta^* a^*$$

However, for a product of Grassmanians,

$$(\theta\eta)^* \equiv \eta^*\theta^* = -\theta^*\eta^*$$

This resembles the rule for hermitian ops.

- Besides being connected by conj., θ & θ^* are independent & to be integrated separately.

The gaussian integral

$$\int d\theta^* d\theta \, e^{-\theta^*\theta} = \int d\theta^* d\theta \, (1-\theta^+\theta) = 1.$$

More generally,

$$\int \prod_i (d\theta_i^* d\theta_i) \, e^{-\sum_{ij}\theta_i^* B_{ij}\theta_j}$$

Applying the eq. $\int d\vec{\theta} f(L\vec{\theta}) = \det L \int d\vec{\theta} f(\vec{\theta})$,

we have

$$\int \prod_i (d\theta_i^* d\theta_i) \, e^{-\sum_i \theta_i^* (B\vec{\theta})_i} = \det B \int \prod_i (d\theta_i^* d\theta_i) \, e^{-\sum_i \theta_i^* \theta_i}$$

$$= \det B$$

In the last step we used

$$e^{f_1(\vec{\theta}) + f_2(\vec{\theta})} = e^{f_1(\vec{\theta})} e^{f_2(\vec{\theta})}$$

which is true only for bosonic functions, because otherwise $[f_1, f_2] \neq 0$.

- Then, using the property under shifts, we can evaluate the integral with sources $\vec{c}$ & $\vec{c}^*$,

$$\int \prod_i d\theta_i^* d\theta_i \; e^{-\sum_{ij} \theta_i^* B_{ij} \theta_j + \sum_i c_i^* \theta_i + \sum_i \theta_i^* c_i}$$

if B is invertible, we do

$$\theta_i \to \theta_i + (B^{-1})_{ij} c_j$$

$$\theta_i^* \to \theta_i^* + (B^{-1})_{ij} c_j^*$$

obtaining the following expr. for the exp:

$$-\theta_i^* B_{ij} \theta_j - c_k^* (B^{-1})_{ki} B_{ij} \theta_j - \theta_i^* B_{ij} B^{-1}_{jk} c_k + c_i^* \theta_i + \theta_i^* c_i$$

$$- c_i^* (B^{-1})_{ij} c_j$$

$$\Rightarrow \int \prod_i d\theta_i^* d\theta_i \; e^{-\sum_{ij} \theta_i^* B_{ij} \theta_j + \sum_i c_i^* \theta_i + \sum_i \theta_i^* c_i} = \det B \; e^{-c^* B^{-1} c}$$

- B needs to be invertible
- If the exponent has to be real,

$$(\theta_i^* B_{ij} \theta_j)^* = \theta_j^* B_{ij}^* \theta_i = \theta_i^* (B^\dagger)_{ij} \theta_j \overset{!}{=} \theta_i^* B_{ij} \theta_j$$

therefore $B^{\dagger} = +B$, hermitian.

- The generating functional for theories with fermions is defined as a path int. over fields that take Grassman vars.

Consider a Dirac field

$$\psi_{\alpha}^{(E)} \,, \quad \bar{\Psi}^{(E),\alpha} \qquad \alpha = 1, \ldots, 4.$$

The generating functional will be

$$Z[\eta, \bar{\eta}] \equiv \int D[\bar{\Psi}^E] D[\psi^E] \cdot \exp\Big\{ -S_E + \int d^4x\, \bar{\eta}^{\alpha}(x) \psi_{\alpha}(x) + \int d^4x\, \bar{\Psi}^{\alpha}(x) \eta_{\alpha}(x) \Big\}$$

We can use it to compute corr. like for bosons, but taking into account anticomm.

$$\frac{\partial}{\delta \eta_{\alpha_2}} \frac{\delta}{\partial \bar{\eta}^{\alpha_1}} Z \Big|_{\substack{\eta=0 \\ \bar{\eta}=0}} = \int D\bar{\Psi} D\psi \; \psi_{\alpha_1}(x_1) \bar{\Psi}^{\alpha_2}(x_2)\, e^{-S_E}$$

$$= \langle 0 | T\{ \psi_{\alpha_1}(x_1) \bar{\Psi}^{\alpha_2}(x_2) \} | 0 \rangle_{Euc.}$$

$$= - \frac{\partial}{\partial \bar{\eta}} \frac{\partial}{\partial \eta} Z \Big|_{\eta\text{'s}=0}$$

$$= - \langle 0 | T\{ \bar{\Psi}^{\alpha_2}(x_2) \psi_{\alpha_1}(x_1) \} | 0 \rangle_{Euc.}$$

which is correct, since time-ordered prod. of two spinors gets a minus sign by flipping two operators.

- The Euclidean action

$$S_E = \left(-iS \ , \ w/ \ \text{"}t \to -i\tau\text{"} \right)$$

$$= -i(-i) \int d^4x_E \left[i\bar{\psi}_E \gamma^0 (i\partial_0^E) \psi_E + i\bar{\psi}_E \gamma^i \partial_i^E \psi_E - m\bar{\psi}\psi \right]$$

$$= \int d^4x_E \left[\bar{\psi}_E g^m \partial_m^E \psi_E + m\bar{\psi}_E \psi_E \right]$$

where we defined "Euclidean" γ matrices

$$g^0 \equiv \gamma^0 \ , \quad g^i \equiv -i\gamma^i$$

$$\Rightarrow \{g^m, g^n\} = 2\delta^{mn} \iff \{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$$

So they define a Clifford algebra of an SO(4) group to be interpreted as the Euclidean Lorentz group.

One can check that the action is real.

- Dirac propagator

Free theory: $S_0 = \int d^4x\, d^4y\, \bar{\Psi}(x) K_{xy} \psi(y)$

with K_{xy} being a 4x4 matrix w/ Dirac indices.

K is

$$K_{xy} = \delta^4(x-y)\, K_y$$

with $K_y = \cancel{\partial} + m$

$$\xrightarrow{\text{Fourier}} \tilde{K}(p) = -i\cancel{p} + m$$

Then the integral

$$Z_0[\eta, \bar{\eta}] = \int D\bar{\Psi} D\psi\, e^{-S_0 + \bar{\eta}\cdot\psi + \bar{\Psi}\cdot\eta}$$

$$= Z[0]\, e^{\bar{\eta}\cdot D\cdot\eta}$$

where

$$D^F_{xy} = \int \frac{d^4p}{(2\pi)^4} e^{-ip(x-y)} \frac{1}{-i\cancel{p}+m} = \langle 0| T\{\psi(x)\bar{\Psi}(y)\}|0\rangle$$

Using $\cancel{p}\cancel{p} = p^2$,

$$\frac{1}{-i\cancel{p}+m} = \frac{i}{\cancel{p}+im} = \frac{i}{p^2+m}(\cancel{p}+im)$$

As for bosons, poles are purely imaginary.

The result is

$$D^F_{xy} = \int \frac{d^4p}{(2\pi)^4} \frac{1}{p^2+m^2} (i\not{p}+m)\, e^{-ip(x-y)}$$

Going to the Euclidean,

$$p_0 \to -ip_0 \quad ; \quad p_i \to p_i$$

$$p^2+m^2 \longrightarrow -(p^2-m^2) \quad ; \quad d^4p \to -i\,d^4p$$

$$\not{p} = p_0 g^0 + p_i g^i \longrightarrow -ip_0 g^0 + p_i g^i = -ip_0\gamma^0 - ip_i\gamma^i$$

$$= -i\not{p}$$

So finally,

$$\Delta^F = \int_{F.C.} \frac{d^4p}{(2\pi)^4} \frac{i}{p^2-m^2} (\not{p}+m)\, e^{-ip\cdot x}$$